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# About a subclass of analytic functions defined by a fractional integral operator

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**Abstract**

In this paper we have introduced and studied the subclass $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ using the fractional integral associated with the convolution product of generalized Sălăgean operator and Ruscheweyh derivative. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$.

**Keywords:** Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, generalized Salagean operator, Ruscheweyh operator.
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## 1 Introduction

Denote by $U$ the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in $U$.

Let $\mathcal{A}(p,l) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+l}^{\infty} a_j z^j, \quad z \in U\}$, with $\mathcal{A}(1,1) = \mathcal{A}$ and $\mathcal{H}[a,l] = \{f \in \mathcal{H}(U) : f(z) = a + a_l z^l + a_{l+1} z^{l+1} + \dots, \quad z \in U\}$, where $p, l \in \mathbb{N}$, $a \in \mathbb{C}$.

**Definition 1.1** *(Al Oboudi [2]) For $f \in \mathcal{A}$, $\alpha \geq 0$ and $m \in \mathbb{N}$, the operator $D_\alpha^m$ is defined by $D_\alpha^m : \mathcal{A} \to \mathcal{A}$ as follows:*

$$\begin{aligned} D_\alpha^0 f(z) &= f(z) \\ D_\alpha^1 f(z) &= (1-\alpha) f(z) + \alpha z f'(z) = D_\alpha f(z) \\ &\dots \\ D_\alpha^m f(z) &= (1-\alpha) D_\alpha^{m-1} f(z) + \alpha z \left(D_\alpha^m f(z)\right)' = D_\alpha \left(D_\alpha^{m-1} f(z)\right), \quad z \in U. \end{aligned}$$

**Remark 1.1** *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D_\alpha^m f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\alpha]^m a_j z^j$, $z \in U$.*

**Remark 1.2** *For $\alpha = 1$ in the above definition we obtain the Sălăgean differential operator [5].*

**Definition 1.2** *(S.T. Ruscheweyh [4]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator $R^n$ is defined by $R^n : \mathcal{A} \to \mathcal{A}$ as follows:*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\dots \\ (n+1) R^{n+1} f(z) &= z \left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

**Remark 1.3** *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j z^j$ for $z \in U$.*

**Definition 1.3** *Let $n, m \in \mathbb{N}$. Denote by $DR_{\alpha}^{m,n}: \mathcal{A} \to \mathcal{A}$ the operator given by the convolution product of the generalized Sălăgean operator $S^m$ and the Ruscheweyh derivative $R^n$:*

$$DR_{\alpha}^{m,n} f(z) = (D_{\alpha}^{m} * R^n) f(z), \tag{1.1}$$

*for any $z \in U$ and each nonnegative integers $m, n$.*

**Remark 1.4** *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $DR_{\alpha}^{m,n} f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\alpha]^m \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j^2 z^j$, $z \in U$.*

**Definition 1.4** *([3]) The fractional integral of order $\lambda$ ($\lambda > 0$) is defined for a function $f$ by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.2}$$

*where $f$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.*

From Definition 1.3 and Definition 1.4 we get the fractional integral associated with the convolution product of generalized Sălăgean operator and Ruscheweyh derivative,

$$D_z^{-\lambda} DR_{\alpha}^{m,n} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{DR_{\alpha}^{m,n} f(t)}{(z-t)^{1-\lambda}} dt =$$

$$\frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt + \sum_{j=2}^{\infty} [1 + (j-1)\alpha]^m \frac{\Gamma(n+j)}{\Gamma(\lambda)\Gamma(n+1)\Gamma(j)} a_j^2 \int_0^z \frac{t^j}{(z-t)^{1-\lambda}} dt,$$

which has the following form, after a simple calculation,

$$D_z^{-\lambda} DR_{\alpha}^{m,n} f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1 + (j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda},$$

for the function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$. We note that $D_z^{-\lambda} DR_{\alpha}^{m,n} f(z) \in \mathcal{A}(\lambda+1, 1)$.

**Remark 1.5** *For $\alpha = 1$ we obtain the operator $D_z^{-\lambda} SR^{m,n}$ defined and studied in [1].*

**Definition 1.5** *Let the function $f \in \mathcal{A}$. Then $f$ is said to be in the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ if it satisfies the following criterion:*

$$\left| \frac{1}{d} \left( \frac{z(D_z^{-\lambda} DR_{\alpha}^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} DR_{\alpha}^{m,n} f(z))''}{(1-\gamma) D_z^{-\lambda} DR_{\alpha}^{m,n} f(z) + \gamma z (D_z^{-\lambda} DR_{\alpha}^{m,n} f(z))'} - 1 \right) \right| < \beta, \tag{1.3}$$

*where $\lambda > 0$, $d \in \mathbb{C} - \{0\}$, $\alpha \geq 0$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $m, n \in \mathbb{N}$, $z \in U$.*

In this paper we shall first deduce a necessary and sufficient condition for a function $f$ to be in the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order $\delta$, $0 \leq \delta < 1$, for these functions.

## 2 Coefficient Inequality

**Theorem 2.1** *Let the function $f \in \mathcal{A}$. Then $f$ is said to be in the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ if and only if*

$$\sum_{j=2}^{\infty} \frac{[1 + (j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda - 2 + \beta|d|) + 1] j + [\gamma(\lambda-1) + 1](\lambda - 1 + \beta|d|) \right\} a_j^2$$

$$\leq (\gamma\lambda + 1)(\beta|d| - \lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}, \tag{2.1}$$

*where $\lambda > 0$, $d \in \mathbb{C} - \{0\}$, $\alpha \geq 0$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $m, n \in \mathbb{N}$, $z \in U$.*

**Proof.** Let $f \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. Assume that inequality (2.1) holds true. Then we find that

$$\left| \frac{z(D_z^{-\lambda} DR_\alpha^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} DR_\alpha^{m,n} f(z))''}{(1-\gamma) D_z^{-\lambda} DR_\alpha^{m,n} f(z) + \gamma z (D_z^{-\lambda} DR_\alpha^{m,n} f(z))'} - 1 \right| =$$

$$\left| \frac{\frac{\lambda(\gamma\lambda+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j \Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [2\gamma(\lambda-1)+1] j + (\lambda-1)[\gamma(\lambda-1)+1] \right\} a_j^2 z^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j \Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 z^{j+\lambda}} \right| \le$$

$$\frac{\frac{\lambda(\alpha\lambda+1)}{\Gamma(\lambda+2)} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j \Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [2\gamma(\lambda-1)+1] j + (\lambda-1)[\gamma(\lambda-1)+1] \right\} a_j^2 \left| z^{j-1} \right|}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)} - \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j \Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 \left| z^{j-1} \right|} \le \beta |d|.$$

Choosing values of $z$ on real axis and letting $z \to 1^-$, we have

$$\sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j \Gamma(n+j)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2$$

$$\le (\gamma\lambda+1)(\beta|d|-\lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}.$$

Conversely, assume that $f \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$, then we get the following inequality

$$Re \left\{ \frac{z(D_z^{-\lambda} DR_\alpha^{m,n} f(z))' + \gamma z^2 (D_z^{-\lambda} DR_\alpha^{m,n} f(z))''}{(1-\gamma) D_z^{-\lambda} DR_\alpha^{m,n} f(z) + \gamma z (D_z^{-\lambda} DR_\alpha^{m,n} f(z))'} - 1 \right\} > -\beta|d|$$

$$Re \left\{ \frac{\frac{\lambda(\gamma\lambda+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [2\gamma(\lambda-1)+1]j + (\lambda-1)[\gamma(\lambda-1)+1] \right\} a_j^2 z^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 z^{j+\lambda}} - 1 + \beta|d| \right\} > 0$$

$$Re \frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1]j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 z^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 z^{j+\lambda}} > 0.$$

Since $Re(-e^{i\theta}) \ge -|e^{i\theta}| = -1$, the above inequality reduces to

$$\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} r^{\lambda+1} - \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1]j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 r^{j+\lambda j}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} r^{\lambda+1} - \sum_{j=2}^{\infty} \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 r^{j+\lambda}} > 0.$$

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.1).
This completes the proof of Theorem 2.1 ∎

**Corollary 2.2** *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. Then*

$$a_j \le \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\}}}, \quad j \ge 2.$$

# 3 Distortion Theorems

**Theorem 3.1** *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. Then for $|z| = r < 1$, we have*

$$r - \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1) \{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}} r^2 \le |f(z)|$$

$$\le r + \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1) \{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}} r^2.$$

*The result is sharp for the function $f$ given by*

$$f(z) = z + \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1) \{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}} z^2, \; z \in U.$$

**Proof.** Given that $f \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$, from the equation (2.1) and since

$$2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}$$

is non decreasing and positive for $j \geq 2$, then we have

$$\sqrt{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}\sum_{j=2}^{\infty} a_j \leq$$

$$\sum_{j=2}^{\infty}\sqrt{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}a_j}$$

$$\leq \sqrt{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}},$$

which is equivalent to,

$$\sum_{j=2}^{\infty} a_j \leq \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}. \tag{3.1}$$

Using (3.1), we obtain

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

$$|f(z)| \leq |z| + \sum_{j=2}^{\infty} a_j |z|^j \leq r + \sum_{j=2}^{\infty} a_j r^j \leq r + r^2 \sum_{j=2}^{\infty} a_j$$

$$\leq r + \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2.$$

Similarly,

$$|f(z)| \geq r - \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2.$$

This completes the proof of Theorem 3.1. ■

**Theorem 3.2** *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. Then for $|z| = r < 1$, we have*

$$-\sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r \leq |f'(z)|$$

$$\leq \sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r.$$

*The result is sharp for the function $f$ given by*

$$f(z) = z + \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}z^2, \ z \in U.$$

**Proof.** From (3.1)

$$f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$$

$$f'(z)| \leq 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1} \leq 1 + \sum_{j=2}^{\infty} j a_j r^{j-1} \leq$$

$$|1+\sqrt{\frac{2\left(\gamma\lambda+1\right)\left(\lambda+2\right)\left(\beta\left|d\right|-\lambda\right)}{\left(1+\alpha\right)^{m}\left(n+1\right)\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}}r.$$

Similarly,

$$|f'(z)|\geq 1-\sqrt{\frac{2\left(\gamma\lambda+1\right)\left(\lambda+2\right)\left(\beta\left|d\right|-\lambda\right)}{\left(1+\alpha\right)^{m}\left(n+1\right)\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}}r.$$

This completes the proof of Theorem 3.2. ■

# 4 Closure Theorems

**Theorem 4.1** *Let the functions $f_k$, $k=1,2,...,l$, defined by*

$$f_k(z)=z+\sum_{j=2}^{\infty}a_{j,k}z^j,\quad a_{j,k}\geq 0,\ z\in U, \tag{4.1}$$

*be in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$. Then the function $h$ defined by*

$$h(z)=\sum_{k=1}^{l}\mu_k f_k(z),\quad \mu_k\geq 0,\ z\in U,$$

*is also in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$, where*

$$\sum_{k=1}^{l}\mu_k=1.$$

**Proof.** We can write

$$h(z)=\sum_{k=1}^{l}\mu_k z+\sum_{k=1}^{l}\sum_{j=2}^{\infty}\mu_k a_{j,k}z^j=z+\sum_{j=2}^{\infty}\sum_{k=1}^{l}\mu_k a_{j,k}z^j.$$

Furthermore, since the functions $f_k$, $k=1,2,...,l$, are in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$, then from Corollary 2.2 we have

$$\sum_{j=2}^{\infty}\sqrt{\frac{\left[1+\left(j-1\right)\alpha\right]^{m}j\Gamma\left(n+j\right)}{\Gamma\left(j+\lambda+1\right)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}a_j}$$

$$\leq\sqrt{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma\left(n+1\right)}{\Gamma\left(\lambda+2\right)}}$$

Thus it is enough to prove that

$$\sum_{j=2}^{\infty}\sqrt{\frac{\left[1+\left(j-1\right)\alpha\right]^{m}j\Gamma\left(n+j\right)}{\Gamma\left(j+\lambda+1\right)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}\left(\sum_{k=1}^{m}\mu_k a_{j,k}\right)=$$

$$\sum_{k=1}^{m}\mu_k\sum_{j=2}^{\infty}\sqrt{\frac{\left[1+\left(j-1\right)\alpha\right]^{m}j\Gamma\left(n+j\right)}{\Gamma\left(j+\lambda+1\right)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}a_{j,k}}$$

$$\leq\sum_{k=1}^{m}\mu_k\sqrt{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma\left(n+1\right)}{\Gamma\left(\lambda+2\right)}}=\sqrt{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma\left(n+1\right)}{\Gamma\left(\lambda+2\right)}}.$$

Hence the proof is complete. ■

**Corollary 4.2** *Let the functions $f_k$, $k=1,2$, defined by (4.1) be in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$. Then the function $h$ defined by*

$$h(z)=(1-\zeta)f_1(z)+\zeta f_2(z),\quad 0\leq\zeta\leq 1,\ z\in U,$$

*is also in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$.*

**Theorem 4.3** *Let*

$$f_1(z) = z,$$

*and*

$$f_j(z) = z + \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}} z^j,$$

$j \geq 2$, $z \in U$.

*Then the function* $f$ *is in the class* $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ *if and only if it can be expressed in the form*

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z), \quad z \in U,$$

*where* $\mu_1 \geq 0$, $\mu_j \geq 0$, $j \geq 2$ *and* $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.

**Proof.** Assume that $f$ can be expressed in the form

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) =$$

$$z + \sum_{j=2}^{\infty} \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}} \mu_j z^j.$$

Thus

$$\sum_{j=2}^{\infty} \sqrt{\frac{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}} \cdot$$

$$\sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}} \mu_j$$

$$= \sum_{j=2}^{\infty} \mu_j = 1 - \mu_1 \leq 1.$$

Hence $f \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$.

Conversely, assume that $f \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$.

Setting

$$\mu_j = \sqrt{\frac{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}} a_j,$$

since

$$\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j.$$

Thus

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z).$$

Hence the proof is complete. ■

**Corollary 4.4** *The extreme points of the class* $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ *are the functions*

$$f_1(z) = z,$$

*and*

$$f_j(z) = z + \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}} z^j,$$

$j \geq 2$, $z \in U$.

## 5 Inclusion and Neighborhood Results

We define the $\delta$- neighborhood of a function $f \in \mathcal{A}$ by

$$N_\delta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta\}. \tag{5.1}$$

In particular, for $e(z) = z$

$$N_\delta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|b_j| \leq \delta\}. \tag{5.2}$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{DR}_{m,n}^{\xi}(\lambda, d, \alpha, \beta, \gamma)$ if there exists a function $h \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ such that

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \xi, \quad z \in U, \quad 0 \leq \xi < 1. \tag{5.3}$$

**Theorem 5.1** *If*

$$\delta = \sqrt{\frac{2\left(\gamma\lambda+1\right)\left(\lambda+2\right)\left(\beta\left|d\right|-\lambda\right)}{\left(1+\alpha\right)^{m}\left(n+1\right)\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}},$$

*then*

$$\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma) \subset N_\delta(e).$$

**Proof.** Let $f \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. Then in view of assertion of Corollary 2.2 and since

$$\frac{\left[1+\left(j-1\right)\alpha\right]^{m} j\Gamma\left(n+j\right)}{\Gamma\left(j+\lambda+1\right)}\left\{\gamma j^{2}+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}$$

$$\geq \frac{\left(1+\alpha\right)^{m}\Gamma\left(n+2\right)}{4\Gamma\left(\lambda+3\right)}\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}$$

for $j \geq 2$, we get

$$\sqrt{\frac{\left(1+\alpha\right)^{m}\Gamma\left(n+2\right)}{4\Gamma\left(\lambda+3\right)}\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}\sum_{j=2}^{\infty}a_j} \leq$$

$$\sum_{j=2}^{\infty}\sqrt{\frac{\left[1+\left(j-1\right)\alpha\right]^{m} j\Gamma\left(n+j\right)}{\Gamma\left(j+\lambda+1\right)}\left\{\gamma j^{2}+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}a_j}$$

$$\leq \sqrt{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma\left(n+1\right)}{\Gamma\left(\lambda+2\right)}},$$

which implise

$$\sum_{j=2}^{\infty}a_j \leq \sqrt{\frac{\left(\gamma\lambda+1\right)\left(\lambda+2\right)\left(\beta\left|d\right|-\lambda\right)}{2\left(1+\alpha\right)^{m}\left(n+1\right)\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}}. \tag{5.4}$$

Applying assertion of Corollary 2.2 in conjunction with (5.4), we obtain

$$\sum_{j=2}^{\infty}ja_j \leq \sqrt{\frac{2\left(\gamma\lambda+1\right)\left(\lambda+2\right)\left(\beta\left|d\right|-\lambda\right)}{\left(1+\alpha\right)^{m}\left(n+1\right)\left\{\left(\lambda+1\right)\left[\gamma\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}} = \delta,$$

by virtue of (5.1), we have $f \in N_\delta(e)$.

This completes the proof of the Theorem 5.1. ■

**Theorem 5.2** *If* $h \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ *and*

$$\xi = 1 + \frac{\delta}{2}\sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}, \tag{5.5}$$

*then*

$$N_\delta(h) \subset \mathcal{DR}^{\xi}_{m,n}(\lambda, d, \alpha, \beta, \gamma).$$

**Proof.** Suppose that $f \in N_\delta(h)$, we then find from (5.1) that

$$\sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta,$$

which readily implies the following coefficient inequality

$$\sum_{j=2}^{\infty} |a_j - b_j| \leq \frac{\delta}{2}. \tag{5.6}$$

Next, since $h \in \mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$ in the view of (5.4), we have

$$\sum_{j=2}^{\infty} b_j \leq \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}. \tag{5.7}$$

Using (5.6) and (5.7), we get

$$\left|\frac{f(z)}{h(z)} - 1\right| \leq \frac{\sum_{j=2}^{\infty}|a_j - b_j|}{1 - \sum_{j=2}^{\infty} b_j} \leq \frac{\delta}{2\left(1 - \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2(1+\alpha)^m (n+1)\{(\lambda+1)[\gamma(\lambda+1+\beta|d|)+1]+\beta|d|\}}}\right)} = 1 - \xi,$$

provided that $\xi$ is given by (5.5), thus by condition (5.3), $f \in \mathcal{DR}^{\xi}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$, where $\xi$ is given by (5.5). ■

# 6 Radii of Starlikeness, Convexity and Close-to-Convexity

**Theorem 6.1** *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{DR}_{m,n}(\lambda, d, \alpha, \beta, \gamma)$. *Then* $f$ *is univalent starlike of order* $\delta$, $0 \leq \delta < 1$, *in* $|z| < r_1$, *where*

$$r_1 = \inf_j \left\{ \frac{(1-\delta)^2 \frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)} \left\{\gamma j^2 + [\gamma(2\lambda - 2 + \beta|d|) + 1] j + [\gamma(\lambda-1)+1](\lambda - 1 + \beta|d|)\right\}}{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2} \right\}^{\frac{1}{2(j-1)}}.$$

*The result is sharp for the function* $f$ *given by*

$$f_j(z) = z + \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\, z^j, \quad j \geq 2.$$

**Proof.** It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \delta, \quad |z| < r_1.$$

Since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)a_j z^{j-1}}{1 + \sum_{j=2}^{\infty} a_j z^{k-1}}|\right| \leq \frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1 - \sum_{j=2}^{\infty} a_j|z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1 - \sum_{j=2}^{\infty} a_j|z|^{j-1}} \leq 1 - \delta.$$

It is equivalent to

$$\sum_{j=2}^{\infty}(j-\delta)a_j|z|^{j-1}\leq 1-\delta,$$

using Theorem 2.1, we obtain

$$|z|\leq\left\{\frac{(1-\delta)^2\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}.$$

Hence the proof is complete. ■

**Theorem 6.2** *Let the function $f\in\mathcal{A}$ be in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$. Then $f$ is univalent convex of order $\delta$, $0\leq\delta\leq 1$, in $|z|<r_2$, where*

$$r_2=\inf_j\left\{\frac{(1-\delta)^2\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}.$$

*The result is sharp for the function $f$ given by*

$$f_j(z)=z+\sqrt{\frac{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}}z^j,\quad j\geq 2. \tag{6.1}$$

**Proof.** It suffices to show that

$$\left|\frac{zf''(z)}{f'(z))}\right|\leq 1-\delta,\qquad |z|<r_2.$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\sum_{j=2}^{\infty}j(j-1)a_jz^{j-1}}{1+\sum_{j=2}^{\infty}ja_jz^{j-1}}\right|\leq\frac{\sum_{j=2}^{\infty}j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}ja_j|z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty}j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}ja_j|z|^{j-1}}\leq 1-\delta,$$

$$\sum_{j=2}^{\infty}j(j-\delta)a_j|z|^{j-1}\leq 1-\delta,$$

using Theorem 2.1, we obtain

$$|z|^{j-1}\leq\frac{(1-\delta)}{j(j-\delta)}\sqrt{\frac{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}},$$

or

$$|z|\leq\left\{\frac{(1-\delta)^2\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}.$$

Hence the proof is complete. ■

**Theorem 6.3** *Let the function $f\in\mathcal{A}$ be in the class $\mathcal{DR}_{m,n}(\lambda,d,\alpha,\beta,\gamma)$. Then $f$ is univalent close-to-convex of order $\delta$, $0\leq\delta<1$, in $|z|<r_3$, where*

$$r_3=\inf_j\left\{\frac{(1-\delta)^2\frac{[1+(j-1)\alpha]^m\Gamma(n+j)}{j^2\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}\right\}^{\frac{1}{2(j-1)}}.$$

*The result is sharp for the function $f$ given by (6.1).*

**Proof.** It suffices to show that

$$|f'(z)-1| \leq 1-\delta, \quad |z| < r_3.$$

Then

$$|f'(z)-1| = \left|\sum_{j=2}^{\infty} j a_j z^{j-1}\right| \leq \sum_{j=2}^{\infty} j a_j |z|^{j-1}.$$

Thus $|f'(z)-1| \leq 1-\delta$ if $\sum_{j=2}^{\infty} \frac{j a_j}{1-\delta}|z|^{j-1} \leq 1$. Using Theorem 2.1, the above inequality holds true if

$$|z|^{j-1} \leq \frac{(1-\delta)}{j}\sqrt{\frac{\frac{[1+(j-1)\alpha]^m j\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta|d|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta|d|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta|d|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}$$

or

$$|z| \leq \left\{\frac{(1-\delta)^2\frac{[1+(j-1)\alpha]^m\Gamma(n+j)}{j^2\Gamma(j+\lambda+1)}\left\{\gamma j^2+\left[\gamma\left(2\lambda-2+\beta|d|\right)+1\right]j+\left[\gamma\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta|d|\right)\right\}}{\left(\gamma\lambda+1\right)\left(\beta|d|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}\right\}^{\frac{1}{2(j-1)}}.$$

Hence the proof is complete. ∎

# References


[1] A. Alb Lupas, *Properties on a subclass of analytic functions defined by a fractional integral operator*, J. Computational Analysis and Applications, Vol. 27, No. 3, 2019, 506-510.

[2] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 27 (2004), 1429-1436.

[3] N.E.Cho, A.M.K. Aouf, *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Tr. J. of Mathematics, Vol. 20, 1996, 553-562.

[4] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[5] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013 (1983), 362-372.